NORMAL STRESSES IN A VISCOELASTIC MEDIUM WITH DECAYING TURBULENT MOTION

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A previous paper [1] dealt with the final stage of degeneration of turbulence in a viscoelastic medium, on the assumption that the stresses (ignoring the hydrostatic pressure) have only tangential components. In other words, that study dealt with liquids and stages in the decay of the motion such that normal stresses may be neglected. Normal stresses are a distinctive feature of an incompressible viscoelastic fluid, and they produce marked effects in laminar flow. It is to be expected that they will produce marked effects on the turbulence of such a liquid.

Such stresses in these fluids are related to nonlinear features of models for liquids, and so those features must be specified if we are to elucidate their effects in turbulence degeneration, and it is necessary to consider a stage of degeneration preceding the final one, when the behavior is still to some degree nonlinear.

§1. First we consider an incompressible liquid whose stress relaxation is described by the Maxwell-Oldroyd equations [2],

$$\frac{\partial\sigma_{ij}}{\partial t}+v_\alpha\frac{\partial\sigma_{ij}}{\partial x_\alpha}-\frac{\partial v_i}{\partial x_\alpha}\sigma_{\alpha j}-\frac{\partial v_j}{\partial x_\alpha}\sigma_{\alpha i}+$$
$$+\frac{1}{\theta}\sigma_{ij}=\frac{\nu}{\theta}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right), \qquad (1.1)$$

and by the equations

$$\frac{\partial v_i}{\partial t}+v_\alpha\frac{\partial v_i}{\partial x_\alpha}=-\frac{\partial p}{\partial x_i}+\frac{\partial\sigma_{i\alpha}}{\partial x_\alpha}, \qquad \frac{\partial v_\alpha}{\partial x_\alpha}=0, \qquad (1.2)$$

in which v_i are the velocity components, σ_{ij} are the components of the kinematic tensor for the stresses after subtracting the isotropic pressure, and p, ν, and θ are the constants of the viscosity and the relaxation times, respectively.

We assume that this liquid contains homogeneous turbulent movements. We choose a coordinate system such that $\langle v_i\rangle = 0$.

Averaging of (1.1) gives

$$\frac{\partial}{\partial t}\langle\sigma_{ij}\rangle+\frac{1}{\theta}\langle\sigma_{ij}\rangle=-\left\langle v_i\frac{\partial\sigma_{\alpha j}}{\partial x_\alpha}\right\rangle-\left\langle v_j\frac{\partial\sigma_{\alpha i}}{\partial x_\alpha}\right\rangle. \quad (1.3)$$

We multiply the equation for v_i in (1.2) by v_j and add to the equation for v_j multiplied by v_i, and then average, to get

$$\frac{\partial}{\partial t}\langle v_i v_j\rangle=\left\langle v_i\frac{\partial\sigma_{\alpha j}}{\partial x_\alpha}\right\rangle+$$
$$+\left\langle v_j\frac{\partial\sigma_{\alpha i}}{\partial x_\alpha}\right\rangle-\left\langle p\frac{\partial v_i}{\partial x_j}\right\rangle-\left\langle p\frac{\partial v_j}{\partial x_i}\right\rangle. \qquad (1.4)$$

We add (1.3) and (1.4), convolute the resulting tensor equation with respect to i and j, and then, since the liquid is incompressible, get the following from (1.2):

$$\frac{\partial}{\partial t}\langle\sigma_{\alpha\alpha}\rangle+\frac{1}{\theta}\langle\sigma_{\alpha\alpha}\rangle=-\frac{\partial}{\partial t}\langle v_\alpha^2\rangle. \qquad (1.5)$$

This equation applies to any stage in the homogeneous turbulent motion of this liquid, and it shows that change in the mean kinetic energy of the turbulence is related to change in the sum of the mean normal stresses, the state of developed turbulence not being stationary.

In fact, if $\langle\sigma_{\alpha\alpha}\rangle\neq 0$, the developed turbulence must be oscillatory, with pulsating mean characteristics (mean kinetic energy and sum of the mean normal stresses $\langle\sigma_{\alpha\alpha}\rangle$).* Let ω be the fundamental frequency of these oscillations; then (1.5) shows that the kinetic energy and normal stress oscillate with the same principal frequency, but with a phase shift π + arc tg$(\theta\omega)^{-1}$. While the energy remains positive, $\langle\sigma_{\alpha\alpha}\rangle$ is sign-varying.

For simplicity, we subsequently consider only the case of isotropic turbulence. Then $\langle\sigma_{ij}\rangle = \sigma\delta_{ij}$ and $\sigma(t) = 1/3\langle\sigma_{\alpha\alpha}\rangle$, so the normal stresses affect mainly the turbulent pressure, which for a viscous liquid consists of the static pressure and the mean dynamic pressure of the pulsations (Reynolds stresses). The pressure for a viscoelastic liquid contains a new component $\sigma(t)$; the stress amplitude σ may be estimated as $\theta\nu v_l^2\times$ $\times\, l^{-2}$, in which v_l and l are the characteristic velocities and length of the turbulent eddies, while for developed turbulence in a viscous liquid the maximum tangential stresses occur when $l\sim\lambda_0$, the internal scale of the turbulence.

We introduce the spectral density $E(k,t)$ of the kinetic energy via

$$\frac{1}{2}\langle v_\alpha^2\rangle=\int\limits_0^\infty E(k,t)\,dk,$$

and for the case of isotropic turbulence we can rewrite (1.5) as

$$\frac{\partial\sigma}{\partial t}+\frac{1}{\theta}\sigma=-\frac{2}{3}\int\limits_0^\infty\frac{\partial E(k,t)}{\partial t}dk. \qquad (1.6)$$

The normal stresses also participate in the interaction between the pulsations of different scales, by virtue of the nonlinear feature of Eqs. (1.1).

*This aspect of the turbulence in a viscoelastic liquid may be related to the observed marked irregularities in the flow of melts and solutions of highly elastic polymers, which have been given names such as "elastic turbulence" [3–5].

We subtract (1.3) from (1.1) to get equations for the pulsating part of the stresses $\sigma_{ij}' = \sigma_{ij} - \langle\sigma_{ij}\rangle$:

$$\frac{\partial\sigma_{ij}'}{\partial t} + v_\alpha \frac{\partial\sigma_{ij}'}{\partial x_\alpha} - \frac{\partial v_i}{\partial x_\alpha}\sigma_{\alpha j}' - \frac{\partial v_j}{\partial x_\alpha}\sigma_{\alpha i}' + \left\langle\frac{\partial v_i}{\partial x_\alpha}\sigma_{\alpha j}'\right\rangle + \left\langle\frac{\partial v_j}{\partial x_\alpha}\sigma_{\alpha i}'\right\rangle + \frac{1}{\theta}\sigma_{ij}' = \frac{\nu + \theta\sigma(t)}{\theta}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \qquad (1.7)$$

This equation differs from the one for the case in which the mean stress σ is absent only in having the viscosity ν replaced by $\nu + \theta\sigma(t)$. It follows from (1.2) and (1.7) that this applies also to the equations for the correlation moments of the distributions of v_i and σ_{ij}', which can be deduced via (1.2) and (1.7); Eqs. (1.6) is then an additional relation linking $\sigma(t)$ to the second correlation moment of the velocity distribution.

Since σ is sign-varying, the expression $\nu \neq \theta\sigma(t)$ that acts as the viscosity in (1.7) may become negative if the amplitude of $\sigma(t)$ exceeds ν/θ. We therefore expect* a change in the character of the turbulence when $\nu/\theta \sim \sigma_{\max} \sim \theta\nu v_l^2 l^{-2}$, i.e., when $\theta\nu_l l^{-1} \sim 1$.

§2. Consider now the degeneration of the turbulence in this liquid. We assume that there is a time when the role of the third-order correlation moments for the velocity and stress has become small but the normal stresses are still important. At this stage, the turbulent moments of various scales (various k) remain coupled, the interaction occurring not via the third-order moments but via the mean stresses $\sigma(t)$.

We use (1.2) and (1.7) to get equations relating the second-order correlation moments of v_i to those of σ_{ij}' (we neglect all third-order moments).** These equations differ from system (1.4) of [1] only in that ν is replaced by $\nu + \theta\sigma(t)$, and these equations allow us to put for $E(k,t)$ that

$$\left[\frac{\partial^2}{\partial t^2} + \frac{2}{\theta}\frac{\partial}{\partial t} + 4k^2\frac{\nu + \theta\sigma}{\theta}\right]\left(\frac{\partial}{\partial t} + \frac{1}{\theta}\right)E = -2k^2\frac{\partial\sigma}{\partial t}E. \qquad (2.1)$$

Equations (2.1) and (1.6) form a closed system of equations for $\sigma(t)$ and $E(k,t)$, and they describe the weak turbulence of a viscoelastic liquid when the third-order correlation functions have become small.

If the $\sigma(t)$ decay fairly rapidly,

$$J_1 = \int_1^\infty |\sigma(t)|\, dt < \infty,$$

and $E(k,t)$ for $t \to \infty$ tends asymptotically [6] to the solution of an equation of a type similar to (2.1) with $\sigma = 0$.

Therefore, this stage for t sufficiently large passes into the final stage [1], while the solution to (2.1) tends to

$$C_1\exp(-t/\theta) + C_2\exp[-t(\beta+1)/\theta] + C_3\exp[t(\beta-1)/\theta],$$

$$\beta(k) = (1 - k^2/k_\theta^2)^{1/2}, \qquad k_\theta = {}^1/_2\, \nu^{-1/2}\theta^{-1/2},$$

$$2C_2 = 2\bar{C}_3 = A(k) + iB(k)$$

for $k > k_\theta$, while the functions $C_i(k)$ are related to E, $\partial E/\partial t$, and $\partial^2E/\partial t^2$ at a certain instant ($t = 0$) in the final stage of degeneration, which is taken as the initial instant, and these can be expressed via the correlation tensors at the same instant:

$$R_{ij}(\mathbf{r}) = \langle v_i(\mathbf{x})\, v_j(\mathbf{x}+\mathbf{r})\rangle,$$

$$S_{ijk}(\mathbf{r}) = \langle v_i(\mathbf{x})\, \sigma_{jk}'(\mathbf{x}+\mathbf{r})\rangle,$$

$$W_{ijkl}(\mathbf{r}) = \langle \sigma_{ij}'(\mathbf{x})\, \sigma_{kl}'(\mathbf{x}+\mathbf{r})\rangle.$$

In the final stage of degeneration, the total kinetic energy of the turbulence is

$$\int_0^\infty E(k,t)\, dk = \int_0^{k_\theta} C_3(k)\exp[t(\beta-1)/\theta]\, dk + \Sigma,$$

in which Σ denotes a group of terms each of which decreases at least as rapidly as $\exp(-t/\theta)$. The main contribution to the integral for $t \gg \theta$ comes from small k, and the asymptotic expression is

$$J_2(k_\theta) = \int_0^{k_\theta} C_3(k)\, e^{-2\nu k^2 t}\, dk. \qquad (2.2)$$

This may be written as $J_2(\infty)$ apart from a term decreasing as $\exp(-t/2\theta)$, i.e., as for a viscous liquid. If $C_0 = \lim k^{-4}C_3(k) < \infty$, $C_0 \neq 0$ for $k \to 0$, as in a viscous liquid (which is a kinematic requirement in essence not dependent on the detailed model as is the incompressible liquid), then for large t the kinetic energy will decrease as $(\nu t)^{-5/2}$.

Then (1.6) gives $\sigma(t)$ as proportional to $\theta\nu^{-5/2}t^{-7/2}$ for $t \gg \theta$, with $J_1 < \infty$.

Consider now the decay of the correlations related to the normal stresses, in particular the pulsating part of the normal stress, with decay of the other components of the correlation tensors in the final stage of degeneracy. It can be shown [see (1.6) in [1]] that

$$S_{i\alpha\alpha}(r) = \langle v_i(\mathbf{x})\, \sigma_{\alpha\alpha}'(\mathbf{x}+\mathbf{r})\rangle = 0 \qquad (2.3)$$

for homogeneous isotropic turbulence of an incompressible medium, i.e., there is no correlation between the velocity distribution and the total normal stresses of the pulsations.

Then we can use the equations for the final stage for the correlation tensors [Eqs. (1.4) of [1]] to show that the correlations in the sum of the normal stresses due to the pulsations,

$$T_{\alpha\alpha}(\mathbf{r}) = \langle \sigma_{\alpha\alpha}'(\mathbf{x})\, p'(\mathbf{x}+\mathbf{r})\rangle,$$

*The value of the dimensionless parameter $\sigma\theta\nu^{-1} = \sigma g^{-1} \sim 1$ ($g = \theta\nu^{-1}$ is the kinematic modulus of elasticity of the liquid) is close [5] to the critical value at which elastic turbulence sets in.

**Since now $\langle\sigma_{ij}\rangle \neq 0$, the correlation tensors have to be written for $\sigma_{ij}' = \sigma_{ij} - \langle\sigma_{ij}\rangle$; the kinematic relations of the type similar to (1.6) of [1] still apply, while $E(k,t)$ is expressed in terms of $R(k,t)$ by $E(k,t) = -4k^4R(k,t)$.

$$W_{ij\alpha\alpha}(\mathbf{r}) = \langle \sigma_{ij}'(\mathbf{x})\,\sigma_{\alpha\alpha}'(\mathbf{x}+\mathbf{r})\rangle,$$

decrease as exp $(-2t/\theta)$ for all r. In particular, $\langle\sigma_{\alpha\alpha}'^2\rangle^{1/2}$ decreases as exp $(-t/\theta)$.

It is readily seen that

$$\frac{\partial^2}{\partial r_\alpha \partial r_\beta} S_{i\alpha\beta}(\mathbf{r}) = 0.$$

This is used with the equation for the final stage of degeneration to give

$$\left(\frac{\partial}{\partial t} + \frac{2}{\theta}\right)\frac{\partial^2}{\partial r_\alpha \partial r_\beta} W_{ij\alpha\beta}(\mathbf{r}) = 0,$$

$$\Delta\langle p'(\mathbf{x})\,p'(\mathbf{x}+\mathbf{r})\rangle = \frac{\partial^2}{\partial r_\alpha \partial r_\beta} T_{\alpha\beta}(\mathbf{r}),$$

$$\Delta T_{ij}(\mathbf{r}) = \frac{\partial^2}{\partial r_\alpha \partial r_\beta} W_{ij\alpha\beta}(\mathbf{r}).$$

These equations imply that the pressure correlations $\langle p'(x)p'(x+r)\rangle$ also decrease as $\exp(-2t/\theta)$.

The components of the correlation tensors decrease in the final stage as for a viscous liquid: the $R_{ij}(r)$ for small r decrease as $t^{-5/2}$; as does the total kinetic energy, while the $W_{ijkl}(r)$ and $S_{ijk}(r)$ for small r decrease as $t^{-7/2}$.

The correlation characteristics related to the normal stresses of the pulsations (including the pressure) thus decrease exponentially in the final stage of degeneration; the other correlations in a viscoelastic liquid decay as in a viscous one. The differences lie in the decay of the mean normal stress (as $t^{-7/2}$)* and in the mode of decay of the small-scale turbulent motions.

§3. In previous sections we have considered a model for viscoelastic liquid with one relaxation time. We now compare the behavior of the normal stresses for other models having other nonlinear features in the equations. The dynamic equations and the equations of incompressibility remain as before [compare (1.2)], and only the defining equations of the (1.1) type are altered.

Consider a model [2] whose defining equations are

$$\frac{\partial \sigma_{ij}}{\partial t} + v_\alpha \frac{\partial \sigma_{ij}}{\partial x_\alpha} + \frac{\partial v_\alpha}{\partial x_i}\sigma_{\alpha j} + \frac{\partial v_\alpha}{\partial x_j}\sigma_{\alpha i} + \frac{1}{\theta}\sigma_{ij} = \frac{\nu}{\theta}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \qquad (3.1)$$

The mean normal stress for homogeneous isotropic turbulence is given by

$$\left(\frac{\partial}{\partial t} + \frac{1}{\theta}\right)\sigma = \frac{1}{3}\frac{\partial}{\partial t}\langle v_\alpha^2\rangle = \frac{2}{3}\int_0^\infty \frac{\partial E(k,t)}{\partial t}\,dk, \qquad (3.2)$$

$$\frac{\partial \sigma_{ij}'}{\partial t} + v_\alpha \frac{\partial \sigma_{ij}'}{\partial x_\alpha} + \frac{1}{\theta}\sigma_{ij}' + \frac{\partial v_\alpha}{\partial x_i}\sigma_{\alpha j}' + \frac{\partial v_\alpha}{\partial x_j}\sigma_{\alpha i}' + \left\langle v_\alpha \frac{\partial \sigma_{\alpha i}'}{\partial x_j}\right\rangle + \left\langle v_\alpha \frac{\partial \sigma_{\alpha j}'}{\partial x_i}\right\rangle = \frac{\nu - \theta\sigma}{\theta}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \qquad (3.3)$$

Equations (3.3) show that here the mean stresses act by replacing the constant viscosity ν by the time-varying one $\nu - \theta\sigma(t)$. At the stage of decay of the turbulent motion where the third-order correlation functions can be neglected, we readily obtain from (1.2) and (3.3) an equation for $E(k, t)$:

$$\left[\frac{\partial^2}{\partial t^2} + \frac{2}{\theta}\frac{\partial}{\partial t} + 4k^2\frac{\nu - \theta\sigma}{\theta}\right]\left(\frac{\partial}{\partial t} + \frac{1}{\theta}\right)E = 2k^2\frac{\partial \sigma}{\partial t}E. \qquad (3.4)$$

Comparison of (3.2) and (3.4) with (1.6) and (2.1) shows that the only difference is in the sign before $\sigma(t)$, so in the final stage, when the total kinetic energy decreases as $At^{-\alpha}$ ($\alpha = 5/2$, $A > 0$), the normal stress will decrease as $5/3\,\theta A t^{-\alpha-1}$ (remaining positive) for the model of (1.1) and as $-5/3\theta A t^{-\alpha-1}$ for the model of (3.1). This sign difference in the final stage could provide the basis for experimental decision between the models.

Next we consider a model whose defining equations explicitly contain nonlinear terms in the stresses:

$$\frac{\partial \sigma_{ij}}{\partial t} + v_\alpha \frac{\partial \sigma_{ij}}{\partial x_\alpha} + \frac{\partial v_\alpha}{\partial x_i}\sigma_{\alpha j} + \frac{\partial v_\alpha}{\partial x_j}\sigma_{\alpha i} + \frac{1}{\theta}\sigma_{ij} - \frac{1}{\nu}\sigma_{i\alpha}\sigma_{\alpha j} = \frac{\nu}{\theta}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \qquad (3.5)$$

Here the role of $\sigma(t)$ does not amount simply to replacing the constant viscosity by a time-varying one even in the case of homogeneous isotropic turbulence. The equation for the pulsating part of the stress becomes

$$\frac{\partial \sigma_{ij}'}{\partial t} + v_\alpha \frac{\partial \sigma_{ij}'}{\partial x_\alpha} + \frac{1}{\theta}\sigma_{ij}' - \frac{2\sigma}{\nu}\sigma_{ij}' + \frac{\partial v_\alpha}{\partial x_i}\sigma_{\alpha j}' + \frac{\partial v_\alpha}{\partial x_j}\sigma_{\alpha i}' - \frac{1}{\nu}\sigma_{i\alpha}'\sigma_{\alpha j}' + \frac{1}{\nu}\langle\sigma_{\alpha i}'\sigma_{\alpha j}'\rangle + \left\langle v_\alpha \frac{\partial \sigma_{\alpha i}'}{\partial x_j}\right\rangle + \left\langle v_\alpha \frac{\partial \sigma_{\alpha j}'}{\partial x_i}\right\rangle = \frac{\nu - \theta\sigma}{\theta}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \qquad (3.6)$$

The equation for the mean stresses is

$$\left(\frac{\partial}{\partial t} + \frac{1}{\theta}\right)\sigma - \frac{1}{\nu}\sigma^2 - \frac{1}{3\nu}\langle\sigma_{\alpha\beta}'^2\rangle = \frac{1}{3}\frac{\partial}{\partial t}\langle v_\alpha^2\rangle, \qquad (3.7)$$

which differs from the (1.6) or (3.2) of the previous models in that it has a nontrivial steady-state solution, with the pulsating part of the stresses related to the normal stress:

$$\sigma_0 - \sigma_0^2/g = {}^1/{}_3 g^{-1}\langle\sigma_{\alpha\beta}'^2\rangle$$

and so $g \geq \sigma_0 > 0$.

The system of equations for the stage of weak turbulence also has the more complicated form

$$\left(\frac{\partial}{\partial t} + \frac{1}{\theta}\right)\sigma - \frac{1}{\nu}\sigma^2 = \frac{2}{3}\int_0^\infty \frac{\partial E(k,t)}{\partial t}\,dk + \frac{1}{3\nu}\langle\sigma_{\alpha\beta}'^2\rangle,$$

$$\left(\frac{\partial}{\partial t} + \frac{2}{\theta} - \frac{4\sigma}{\nu}\right)\langle\sigma_{\alpha\beta}'^2\rangle = -4\frac{\nu - \theta\sigma}{\theta}\int_0^\infty \frac{\partial E(k,t)}{\partial t}\,dk,$$

$$\left(\frac{\partial}{\partial t} + \frac{2}{\theta} - \frac{4\sigma}{\nu}\right)\left(\frac{\partial}{\partial t} + \frac{1}{\theta} - \frac{2\sigma}{\nu}\right)\frac{\partial}{\partial t}E(k,t) = 2k^2\frac{\partial \sigma}{\partial t}E(k,t) - 4k^2\frac{\nu - \theta\sigma}{\theta}\left(\frac{\partial}{\partial t} + \frac{1}{\theta} - \frac{2\sigma}{\nu}\right)E(k,t). \qquad (3.8)$$

In the final stage, where the mean normal stresses have become small, they decay as $t^{-7/2}$ and have a negative sign, as in the previous model.

*There are at least two difficulties in testing this by experiment: measurement of small normal stresses and distinction of Reynolds stresses. Nonisotropic inhomogeneous turbulence (e.g., for a flow with transverse shear) here has advantages.

The following are models that lack a stage of weak turbulence (in the sense of §2).

First we have the case $\sigma_{\alpha\alpha} = 0$. A mean normal stress $\langle\sigma_{ij}\rangle = \sigma\delta_{ij}$ is absent in the case of isotropic turbulence for this.

Another model has relaxation in the deformation rate, e. g., a second-order Rivlin-Ericksen liquid [7] whose defining equations are

$$\sigma_{ij} = \nu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \nu\theta\left(\frac{\partial a_i}{\partial x_j} + \frac{\partial a_j}{\partial x_i} + 2\frac{\partial v_\alpha}{\partial x_i}\frac{\partial v_\alpha}{\partial x_j}\right) +$$

$$+ \nu_c\left(\frac{\partial v_\alpha}{\partial x_i} + \frac{\partial v_i}{\partial x_\alpha}\right)\left(\frac{\partial v_\alpha}{\partial x_j} + \frac{\partial v_j}{\partial x_\alpha}\right), \quad a_i = \frac{\partial v_i}{\partial t} + v_\alpha\frac{\partial v_i}{\partial x_\alpha}, \quad (3.9)$$

in which ν, θ, and ν_c are constants of the material in the simplest case.

We average this equation to get the mean stress for homogeneous isotropic turbulence as

$$\sigma = -{}^2/_3(\nu\theta + \nu_c)\lim_{r\to 0}\Delta R_{\alpha\alpha}(r) =$$

$$= -{}^4/_3(\nu\theta + \nu_c)\int_0^\infty E(k, t)\,k^2 dk\,. \quad (3.10)$$

In the final stage, in which the decay of motions with very small k is of the viscous type [1], the total kinetic energy decreases as $At^{-5/2}$, while the mean stress decreases as $-5/3(\theta + \nu_c/\nu)At^{-7/2}$.

Thus $\sigma(t)$ in the final stage decreases as $t^{-7/2}$, in spite of the difference between the defining equations (the Reynolds stresses decrease only as $t^{-5/2}$).

In §§2 and 3 we have considered in detail only the final stage, when the behavior of the viscoelastic liquid is generally similar to that of a viscous one. Now we consider the preceding stage, which is described by (1.6) and (2.1), (3.2) and (3.4), (3.8). These equations have been deduced on the assumption that the third-order correlation functions can be neglected while the mean normal stresses are retained, and they describe a stage in the decay called that of weak turbulence, when the generation of motions with new spatial scales can be neglected. In fact, the energy initially localized in some region of wave space is indicated by (1.6) and (2.1) (the only ones to be used, for brevity) as remaining in that space throughout all subsequent time. The nonlinearity in the system of equations affects only E(k, t) and σ(t) as functions of time, which results from the interaction between movements on different scales.

§4. Consider the behavior of E(k,t) at small t, when the nonlinear features of (1.6) and (2.1) are still important. We consider t so small that the deviations $\sigma(t) - \sigma_0$ and $E'(k,t) = E(k,t) - E(k,0)$ from the initial distributions can be considered as small, and the equations may be linearized:

$$\left[\frac{\partial^2}{\partial t^2} + \frac{2}{\theta}\frac{\partial}{\partial t} + 4k^2\frac{\nu + \theta\sigma_0}{\theta}\right]\left(\frac{\partial}{\partial t} + \frac{1}{\theta}\right)E'(k, t) =$$

$$= -2k^2 E(k, 0)\left(\frac{\partial}{\partial t} + \frac{2}{\theta}\right)\sigma\,,$$

$$\left(\frac{\partial}{\partial t} + \frac{1}{\theta}\right)\sigma(t) = -\frac{2}{3}\int_0^\infty \frac{\partial E'(k, t)}{\partial t}dk\,. \quad (4.1)$$

We eliminate $\sigma(t)$ to get a single equation for $E'(k,t)$:

$$\left[\frac{\partial^2}{\partial t^2} + \frac{2}{\theta}\frac{\partial}{\partial t} + 4k^2\frac{\nu + \theta\sigma_0}{\theta}\right]\left(\frac{\partial}{\partial t} + \frac{1}{\theta}\right)^2 E'(k, t) =$$

$$= \frac{4}{3}k^2 E(k, 0)\left(\frac{\partial^2}{\partial t^2} + \frac{2}{\theta}\frac{\partial}{\partial t}\right)\int_0^\infty E'(p, t)\,dp\,. \quad (4.2)$$

First we consider the decay of eddies of some given scale $\sim 1/k_1$, which are described by the spectral function $E(k,t) = E_1(t)\delta(k - k_1)$, in which $E_1(t = 0) = E_{10}$ defines the initial energy.

Equation (4.2) for $E_1'(t) = E_1(t) - E_{10}$ becomes

$$\left[\frac{\partial^2}{\partial t^2} + \frac{2}{\theta}\frac{\partial}{\partial t} + 4k_1^2\frac{\nu + \theta\sigma_0}{\theta}\right]\left(\frac{\partial}{\partial t} + \frac{1}{\theta}\right)^2 E_1' =$$

$$= \frac{4}{3}k_1^2 E_{10}\left(\frac{\partial^2}{\partial t^2} + \frac{2}{\theta}\frac{\partial}{\partial t}\right)E_1'\,. \quad (4.3)$$

The solution is

$$E_1'(t) = \sum_{i=1}^4 C_i \exp\frac{-t(1 + p_i)}{\theta}\,,$$

$$E_1'(0) = \sum_{i=1}^4 C_i = 0\,,$$

in which the C_i are constants defined by the initial conditions for E(t) and σ(t), while the p_i are the roots of the quadratic equations

$$2(p^2 - 1) = \left[4\theta^2 k_1^2\left(\frac{E_{10}}{3} - \frac{\nu + \theta\sigma_0}{\theta}\right) - 1\right] \pm$$

$$\pm\left\{\left[4\theta^2 k_1^2\left(\frac{E_{10}}{3} - \frac{\nu + \theta\sigma_0}{\theta}\right) - 1\right]^2 -\right.$$

$$\left. - 16\theta k_1^2(\nu + \theta\sigma_0)\right\}^{1/2}. \quad (4.4)$$

These equations show that for $\nu + \theta\sigma_0 < 0$ the equation with the plus sign before the root always leads to $p^2 > 1$, so one of the solutions $\exp[-t(1 + p)/\theta]$ will be an increasing one. If $\nu + \theta\sigma_0 > 0$, Re $p^2 > 1$ will be obeyed if $\theta^{-2}k_1^{-2}/4 < E_{10}/3 - (\nu + \theta\sigma_0)/\theta$, so (4.3) has solutions that increase with time for certain initial conditions.

In what follows we consider only the case $(\nu + \theta\sigma_0)/\theta > 2E_0/3$, whereupon the above argument show that all the solutions to (4.3) will be decaying ones. If k_1 is small ($k_1^2\theta(\nu + \theta\sigma_0) \ll 1$), (4.4) gives

$$p_{1,2}^2 \approx 1 - 4\theta k_1^2(\nu + \theta\sigma_0), \qquad p_{3,4}^2 \approx {}^4/_3\theta^2 k_1^2 E_{10}\,. \quad (4.5)$$

Such expressions describe the decay of a single large-scale eddy. Now we consider how the decay of this eddy is affected by the presence of an eddy of a different scale. For this purpose we take the spectral function as $E(k,t) = E_1(t)\delta(k - k_1) + E_2(t)\delta(k - k_2)$, whereupon (4.2) takes the form of a system of two equations with constant coefficients for the two functions $E_1'(t) = E_1(t) - E_{10}$ and $E_2'(t) = E_2(t) - E_{20}$:

$$\left[\frac{\partial^2}{\partial t^2} + \frac{2}{\theta}\frac{\partial}{\partial t} + 4k_1^2\frac{\nu + \theta\sigma_0}{\theta}\right]\left(\frac{\partial}{\partial t} + \frac{1}{\theta}\right)^2 E_1' =$$

$$= \frac{4}{3}k_1^2 E_{10}\left(\frac{\partial^2}{\partial t^2} + \frac{2}{\theta}\frac{\partial}{\partial t}\right)(E_1' + E_2')\,,$$

$$\left[\frac{\partial^2}{\partial t^2} + \frac{2}{\theta}\frac{\partial}{\partial t} + 4k_2^2\frac{\nu + \theta\sigma_0}{\theta}\right]\left(\frac{\partial}{\partial t} + \frac{1}{\theta}\right)^2 E_2' =$$

$$= \frac{4}{3}k_2^2 E_{20}\left(\frac{\partial^2}{\partial t^2} + \frac{2}{\theta}\frac{\partial}{\partial t}\right)(E_2' + E_1')\,. \quad (4.6)$$

If the solution is to have a time dependence of the form $\exp[-t(1 + p)/\theta]$, we must have that p^2 satisfies

$$p^4\,[p^2 - 1 + 4k_1^2\theta(\nu + \theta\sigma_0)] \times$$

$$\times [p^2 - 1 + 4k_2^2\theta(\nu + \theta\sigma_0)] =$$
$$= {}^4/_3\,\theta^2 p^2 (p^2 - 1)\,[(p^2 - 1)(k_1^2 E_{10} + k_2^2 E_{20}) +$$
$$+ 4k_1^2 k_2^2 (\nu + \theta\sigma_0)\theta (E_{10} + E_{20})]\,. \qquad (4.7)$$

Here, as previously, we write out the values of p^2 only for small k_1 and k_2:

$$\theta, {}^4/_3\,\theta^2 (E_{10}k_1^2 + E_{20}k_2^2) + \ldots,$$
$$1 - 4\theta(\nu + \theta\sigma_0)k_1^2 + \ldots,$$
$$1 - 4\theta(\nu + \theta\sigma_0)k_2^2 + \ldots. \qquad (4.8)$$

Comparison of (4.8) with (4.5) shows that the set of p is very different, i.e., the time dependence of $E_1^i(t)$ is different. The marked effect of the mean normal stress on the rate of degeneration can be understood from the general form of the equations. The viscosity begins to play the leading part in the late stage, and the normal stresses in the above models influence primarily the viscosity [see (1.7) and (3.3)], so the normal stresses in a viscoelastic liquid greatly affect not only the small-scale movements but also the large-scale ones, in any case in the stage of weak turbulence.

Finally we consider the role of the mean normal stresses in the undamped homogeneous isotropic turbulence of a viscoelastic liquid. An obvious effect of the elasticity is on the small-scale (high-frequency) eddies, if the characteristic length $(\nu\theta)^{1/2}$ becomes comparable with the size of the eddy. The effects on larger eddies are less obvious, though we expect these eddies to be affected by normal stresses. The mean normal stress produces coupling between turbulent movements on all scales, as all eddies contribute to the normal stress, while that stress appears in the equation describing the behavior of the eddies. If the smallest eddies had the highest normal stresses, changes in the small-scale motions would act via those stresses on the large-scale ones, and would affect, in particular, transport phenomena.

This can occur in liquids of low elasticity. The normal stress is proportional to the square of the velocity gradient in laminar flow in such a liquid. The highest velocity gradients occur in the smallest eddies in a viscous liquid (regions of dissipation), and the same will apply for an elastic liquid closely resembling a viscous one. The normal stresses influence the large energy-bearing eddies and so influence the momentum transfer.

This effect of the normal stresses may be the reason why small amounts of polymers reduce the resistance in turbulent flow [8].

Note. Neglect of the nonlinear terms in the dynamic equation implies that the Reynolds numbers are small, i.e., $v_l/l \ll \nu/l^2$.

The quasi-linear stage of weak turbulence is meaningful for motion involving fluctuations in the tangential and normal stresses, $\nu v_l/l$, $\nu\theta v_l^2/l^2 \ll \sigma$.

It is possible to estimate σ on the basis that the highest velocity gradients near the terminal laminar stage will occur in movements whose scale l_0 is that where the energy fluctuations are largest at that instant, so $\sigma \sim \nu\theta v_{l0}^2/l_0^2$. All three estimates show that the stage of weak turbulence is meaningful only for motions that do not bear most of the energy (the larger and smaller ones).

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